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"Certainty and Uncertainty in the Taxation of Risky Returns"

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Northwestern Law School

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NYU TAX POLICY COLLOQUIUM

(All sessions meet Thursday 4:00-6:00 pm, Furman-120, NYU Law School)

1. January 15 – Daniel Shaviro, NYU Law School. "The Long-Term Fiscal Gap: Is the Main Problem Generational Inequity?"

2. January 22 – Alan Auerbach, Berkeley Economics Department and NYU Law School. "Understanding U.S. Corporate Tax Losses." http://www.nber.org/papers/w14405.pdf

3. January 29 – Edward Kleinbard, Joint Committee on Taxation. "A Reconsideration of Tax Expenditure Analysis."

4. February 5 – Amy Finkelstein, MIT Economics Department, "EZ-Tax: Tax Salience and Tax Rates."

5. February 12 – Dorothy Brown, Emory Law School. "Shades of the American Dream."

6. February 19 – Yoram Margalioth, Tel Aviv University Law School and NYU Law School. "Employing Statistical Stigma As a Welfare Ordeal."

7. February 26 – Leslie McCall, Northwestern University Sociology Department. "Americans' Social Policy Preferences in the Era of Rising Inequality."

8. March 5 – Michael Doran, University of Virginia Law School. "Managers, Shareholders, and the Corporate Double Tax."

9. March 12 – David Duff, University of Toronto Law School. "Tax Fairness and the Tax Mix."

10. March 26 – Emmanuel Saez, Berkeley Economics Department. "Details Matter: The Impact of Presentation and Information on the Take-Up of Financial Incentives for Retirement Saving."

11. April 2 – Lily Batchelder, NYU Law School. "Savings Incentives with Insurance Objectives: A Bankrupt Approach?"

12. April 9 – Mihir Desai, Harvard Business School and NYU Law School. "Investor Taxation in Open Economies."

13. April 16 – Mitchell Kane, NYU Law School. "Taxation and Global Cap and Trade."

14. **April 23 – Thomas Brennan, Northwestern Law School. "Certainty and Uncertainty in the Taxation of Risky Returns."**

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Certainty and Uncertainty in the Taxation of Risky Returns

Thomas J. Brennan*

This Draft: April 6, 2009

Abstract

Existing research has shown that a proportionate tax on income is equivalent to a tax on the risk-free rate of return applied to capital. What is not so well understood is the nature of non-proportionate taxes on income, such as those having loss disallowances or progressivity features. I extend the general equilibrium techniques traditionally applied to proportionate taxes to gain better understanding of more realistic non-proportionate taxes. I frame the question of how to go about this extension by analogizing proportionate taxes to financial forwards and more general taxes to structured financial options, and I find that option pricing theory and methods carry over naturally. In general, I find that the burden of an income tax is akin to the price of a corresponding option and that non-proportionate income taxes generally burden risky returns just as option prices generally reflect the volatility inherent in risky assets. I develop the theory in most detail in the case of binomial asset returns, and I particularly focus on the example of a tax that is proportionate for gains but allows no deduction for losses. The binomial context and this example are illustrative of the general ideas and serve to show the degree to which risky returns are burdened by even a simple deviation from strict proportionality. I also go on to lay out the ideas and structure for extending the analysis to very general tax systems and to continuous, rather than binomial, risky returns. The results obtained show broadly that non-proportionate income taxes burden risky returns on assets in precise, systematic and quantifiable ways. This new theoretical result calls for a revision of the traditional understanding of what an income tax is able to, and does, accomplish in terms of burdening risky components of return.

Keywords: Income Taxation; Proportionate Income Taxation; Risky Return to Assets; Domar-Musgrave; General Equilibrium

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1 Overview

It is well understood that a constant-rate tax levied proportionately on investment income is equivalent to a tax on the risk-free return to initial investment wealth, at least under idealized assumptions about the supply of assets, the interest rate for borrowed money, and the de minimis nature of transaction costs.[1] What is not so well understood is how precisely this result changes when the tax is not proportionate.[2] This paper addresses the gap in understanding by using the same basic tools previously used to analyze constant-rate proportionate taxes and extending them to allow for systematic analysis of more general systems of taxation. We find a number of interesting and important results, the most important of which is the fact that non-proportionate income taxes generally burden the risky component of returns in a precise, systematic and quantifiable way. This stands in contrast to the case of constant-rate proportionate taxes and calls for a revision in the understanding of what an income tax on investments is able to, and does, accomplish.

The analysis of proportionate taxes dates back to the classic results of Domar and Musgrave,[3] and it has been broadly extended and placed in a general equilibrium setting.[4] The fundamental underlying idea of all this analysis can be expressed in the language of finance in terms of a forward contract on a risky asset. For every unit of risky asset a taxpayer holds, his future obligation to pay taxes is the same as a short position in t units of a forward contract on the asset, where t is the constant tax rate.[5] The government, on the other hand, has a right to collect taxes paid that is the same as the long forward position complementary to the short position of the taxpayer. The short and long forward positions can be perfectly hedged by offsetting long and short positions, respectively, in the risky asset, and the cost of this hedging depends only on the risk-free rate of return and the initial value of risky asset held.[6] As a result, the proportionate tax can be costlessly transformed into a tax depending only on the risk-free return to capital.

[1] Domar and Musgrave (1944, 1946) lay out the result in its original form, and Kaplow (1991, 1994) generalizes it to a general equilibrium setting. Additional discussion and development in the legal literature can be found, for example, in Warren (1996) and Weisbach (2004, 2005). See Avi-Yonah (2004) for a discussion of some of the limitations resulting from borrowing costs and transaction costs.

[2] The literature has taken note of the fact that the usual arguments do not handle the case of non-proportionate taxes. See Schenk (2000) and Zelenak (2006) for very good discussions and analysis of the limitations in this regard. Notwithstanding the acknowledgement of the issue and the work in the literature, little has been done by way of analyzing non-proportionate taxes using the sophisticated methods used to understand proportionate taxes. An exception is Weisbach (2004), who uses general equilibrium methods similar to those in Kaplow (1991, 1994). However, that analysis is focussed on multiple tax rates for multiple assets rather than non-proportionality applied to a single asset, the topic studied in the current paper.

[3] See Domar and Musgrave (1944, 1945).

[4] See Kaplow (1991, 1994), who innovates and broadens the analysis by incorporating government action into the equilibrium and thereby obtains results that are truly of a general equilibrium character.

[5] By a forward contract, or more simply forward, we mean a contract under which one party sells and the other party buys an asset at a specified price at a specified exercise time. The long position is the obligation to purchase and the short position is the obligation to sell. We do not distinguish between the terms "forward contract" and "future contract" but instead use the former term to stand for any contract for an agreement to buy and sell at a specified future price and time.

[6] For example, the risk in a short forward on a single share of stock can be perfectly hedged by a long position in the share of stock. If both of these positions are entered into today, then at the time of exercise of the forward, the obligation under the forward to deliver a share of stock can be exactly satisfied with the share of stock owned. The price of the forward today is the price of the stock, adjusted by the risk-free rate,

If a system of taxation is not proportionate, however, the usual analysis no longer applies since the obligation to pay tax and the right to collect tax are no longer the same as simple forward positions in the risky asset. For example, if tax is levied on gains but no offsetting deduction is allowed for losses, then the taxpayer is short a call option on the risky asset while the government has the complementary long position in the same option. It is still possible to take positions in the risky asset that hedge the option position represented by the tax,[7] but the cost of hedging the option depends not only on the risk-free rate but also on the risk inherent in the underlying risky asset.[8]

A tax system can fail to be proportionate for a variety of reasons. As discussed in the preceding paragraph, it may not allow for loss-offsetting deductions. In fact, any progressive tax system also fails to be proportionate, since different levels of return are taxed at different rates. Moreover, a system that allows taxpayers discretion about the timing or rate of tax applicable to returns also fails to be proportionate. A common example of this is the choice taxpayers have about when to realize gains and losses on capital asset – they may often choose, perhaps, to realize losses sooner and gains later. Another example is the ability of U.S. firms to decide when to keep earnings of a foreign subsidiary permanently reinvested abroad – they may often choose, perhaps, to permanently reinvest earnings in foreign jurisdictions with low tax rates so as to avoid additional tax upon repatriation of earnings.

Each of the foregoing examples of non-proportionate taxes can be expressed as a deterministic function of the stochastic return on the risky assets in question. That is to say, even though risky asset returns are uncertain at the start, once the return is known, there is a definite and known way in which the tax on the return can be computed. This computation may involve choice on the part of the taxpayer, but if the taxpayer's preferences are known from the start, then the choice he will make given any particular levels of return is also known from the start.

If the tax on a risky asset is deterministic, conditioned upon knowing the asset return level, then the tax payable can, in general, be hedged with a (possibly dynamic) position in the underlying asset. The tax can be thought of as a complex option on the risky asset, and the cost of this option can be determined by option pricing techniques that compute the cost of the hedging strategy in present value terms.[9] When the tax is not proportionate, the price

and this adjustment is the cost of forming the hedge. See Bodie, Kane and Marcus (2005) for further details and discussion of forward hedging and pricing.

[7]The hedging required will generally evolve dynamically. At each point in time, a particular position in the risky asset will hedge the option position, but the size of the required hedge will change over time depending upon how the asset value changes.

[8]The present value of the cost of hedging an option is the same as the price of the option. The techniques that can be used to determine this price rely on replicating the option dynamically using hedging positions in the asset and then evaluating the present value cost of such a dynamic hedging strategy. This cost generally depends on the amount of risk inherent in the underlying asset, as well as other factors. For example, if the returns of the risky asset are log-normally distributed, then the Black-Scholes formula can be applied to determine the option price, and if the risk-free rate is zero and the asset pays no dividends, an at-the-money all option costs $\frac{1}{\sqrt{2\pi}} S_0 \sigma \sqrt{t}$, where S_0 is the current price of the asset in dollars, σ is the volatility (risk) of the stock, and t is the time to expiration of the option. Since $1/\sqrt{2\pi} \approx 0.40$, this is approximately equal to 40% of the volatility of the stock in dollars over the term of the option. See Bodie, Kane and Marcus (2005) for further details and discussion of option hedging and pricing.

[9]There is a slight additional wrinkle in that the hedging position itself will in general affect the overall

of this option, and hence the burden imposed by the tax, will generally reflect the volatility, or risk, of the underlying asset in a specific way, meaning that a non-proportionate income tax does indeed burden risky returns, and that this burden is knowable and understandable.

In order to implement the foregoing discussion and ideas, we need a conceptual framework which will allow us to model various systems of taxation. We take as our starting place the general equilibrium model introduced by Kaplow (1991, 1994), and we extend it so as to allow us to express different systems of taxation in compact notation as well as to define in general what it means for two systems of taxation to be equivalent. We develop this model in Section 2. The ideas and details are very much based in Kaplow's work, but it is useful to recast the results and approach anew in a way that allows us to use them most effectively for the analysis that follows in Section 3.

In Section 3, we apply the model developed in Section 2 to non-proportionate systems of taxation using the principles developed in this Overview. We consider first non-proportionate taxes when there is one risky asset that has only a binary choice possibility for its return values. The reason for restricting to the special case of binary asset returns is that it allows us to use the general equilibrium framework to study very general systems of taxation without any additional assumptions. Moreover, the binary approach mirrors the binary model for option pricing and lays the groundwork for studying continuous return assets in a multi-period setting. We do not develop a multi-period model in this paper, but we do discuss what is involved in extending to more general settings.

Finally, in Section 4, we discuss our findings and conclude.

tax, but this presents only a small complication and can be accounted for when structuring the appropriate hedging positions and calculating the corresponding option price.

2 Extending the Usual Model

In this section, we describe and extend the general equilibrium analysis based on the principles of Domar and Musgrave and developed by Kaplow.[10] The goal is to provide background and introduce the foundation upon which the results of Section 3 are based. In 2.1, the economic model upon which the analysis relies is detailed. In 2.2, the notion of equivalence between systems of taxation under the model is defined. In 2.3, the well-known equivalence between a proportionate tax and a tax on risk-free return to wealth is derived. In 2.4, we provide a numerical example to illustrate the equivalence between a proportionate tax and a tax on risk-free return to wealth.

2.1 Economic Model and Description of Equilibrium

We consider a simplified economic model for the world consisting of only two points in time, $\tau = 0$ and $\tau = 1$. At the first point in time, individuals make decisions about their allocation between labor and leisure. They are paid up-front for their labor, and they make decisions about how much of this wage to consume at time $\tau = 0$ and how much to save until time $\tau = 1$. They also decide how to allocate their savings portfolio between a risky asset and a risk-free asset, the only two investment assets available in this simple model.

In addition to individuals, there is a special additional actor, the government. The government makes an investment decision at time $\tau = 0$ by deciding how to allocate the wealth it saves between the same risky asset and risk-free asset that are available to other investors. The government also collects an amount of tax based on investment returns from each individual at time $\tau = 1$. The amount of tax follows a prescribed formula and may depend both on initial wealth and on returns to investments. We don't specify the particular nature of the tax at this point because we want to consider various different possible regimes in what follows.

Under standard conditions for the relevant utility functions, there will be a set of choices by the individuals and the government that results in a stable equilibrium. When this set of choices is implemented, no individual or the government will, on the margin, have incentive to alter their choices because doing so would not enhance their utility. The equilibrium set of choices includes the decisions about labor and leisure, the decisions about consumption and savings, and also the decisions about investment portfolio allocation. For purposes of this paper, we focus only on the last of these decisions, namely the one involving investment portfolio allocation.

The set of portfolio allocation choices at time $\tau = 0$ can be represented compactly by the matrices

$$\mathbf{P} = \begin{bmatrix} \mathsf{a} & \mathsf{b} \end{bmatrix} \quad \text{and} \quad \mathbf{P}_G = \begin{bmatrix} a_G & b_G \end{bmatrix}, \tag{1}$$

where a and b are vectors with elements equal to the positions investors take in the risky asset and the risk-free asset, respectively, and where a_G and b_G are the positions the government takes in these assets.

The utility realized by an investor depends upon the final investment wealth he holds, after taking into account the effect of taxes. This final investment wealth can be expressed in terms of $\mathbf{P}$ and the returns on the risky and risk-free assets. For the return on the risky

[10] See Kaplow (1991, 1994).

asset, we write x. It is a quantity that is not certain until time $\tau = 1$. For the return on the risk-free asset, we write r_f. The expression for after-tax investment wealth is

$$\mathbf{W}(\mathbf{P}; \mathbf{T}) = \mathbf{P}(\mathbf{1} + \mathbf{r}) - \mathbf{T}(\mathbf{P}, \mathbf{r}), \tag{2}$$

where $\mathbf{1}$ is a 2-dimensional column vector of ones,[11] where $\mathbf{r}$ is the 2-dimensional column vector of asset returns,

$$\mathbf{r} = \begin{bmatrix} x \\ r_f \end{bmatrix}, \tag{3}$$

and where $\mathbf{T}(\mathbf{P}, \mathbf{r})$ is a vector that makes appropriate adjustment for taxes paid by investors.

The expression in Equation (2) is central to the analysis that follows because it represents the final wealth amounts resulting from portfolio allocation decisions, and these amounts are what investors take into consideration when determining their optimal equilibrium portfolio choice. In Section 2.2, we consider the effect of changes in the system of taxation represented by $\mathbf{T}$, and we define what it means for two systems of taxation to be equivalent.

2.2 Equivalent Systems of Taxation

We consider two systems of taxation of investments with the goal of comparing them and defining what it means for them to be equivalent. Each system can be represented by a function $\mathbf{T} = \mathbf{T}(\mathbf{P}, \mathbf{r})$ of the type described in Section 2.1. This function produces a vector with elements equal to the amount of taxes paid by investors depending on the portfolio allocation choices represented in $\mathbf{P}$ and the asset returns represented in $\mathbf{r}$. To distinguish between the functions for the two systems, we write the first as $\mathbf{T}$ and the second as $\mathbf{T}'$.

We define the second system of taxation to be "at least as good as" the first if, for any choices of investor and government portfolio allocations, $\mathbf{P}$ and $\mathbf{P}_G$, there are adjusted choices, $\mathbf{P}'$ and $\mathbf{P}'_G$, such that the following three conditions are satisfied:

- A(i) the adjustment has no net cost to any investor or the government, so that the initial wealth amounts of each investor and of the government remain the same;
- A(ii) the adjustment by the government is the negative of the aggregate adjustment by the investors, in the sense that the government adjustment consists of long and short positions that exactly offset the short and long positions, respectively, comprising the aggregate investor adjustment; and
- A(iii) the final after-tax wealth amounts for all investors produced by original portfolio choices under the first system of taxation is the same as that produced by the adjusted portfolio choices under the second system of taxation.

If these conditions are satisfied, then the adjustment does not change the initial wealth of any investor or of the government.[12] Moreover, net positions held in each asset are

[11]Note that in the remainder of the paper $\mathbf{1}$ will always represent a column vector of all ones, but that the dimensionality of this vector may be different depending upon the situation.

[12]This is true for individual investors because of Condition A(i). It is true for the government because the change in the government's wealth is the aggregate of the changes in individual investor wealth.

unchanged.[13] In addition, the adjustment does not change the final after-tax wealth amount of any individual investor or of the government.[14] Thus, any set of portfolio choices under the first system of taxation corresponds to a set of portfolio choices under the second system that produces identical results both for individual investors and for the government. It is therefore natural to say that the second system is at least as good as the first when the conditions are met.

We define the two systems of taxation to be equivalent if the second is at least as good as the first and if the following additional condition holds:

B(i) each set of portfolio allocation choices, $\mathbf{P}'$ and $\mathbf{P}'_G$, is equal to the adjusted portfolio of some other set of portfolio allocation choices, $\mathbf{P}$ and $\mathbf{P}_G$, for which Conditions A(i), A(ii), and A(iii) are satisfied.

If this condition holds, then each set of portfolio choices, $\mathbf{P}'$ and $\mathbf{P}'_G$, corresponds to an adjusted set of portfolio choices, $\mathbf{P}$ and $\mathbf{P}_G$, such that the adjustment has no cost and does not affect asset values, and such that the original choices produce the same after-tax wealth amounts under the first system as the adjusted choices produce uner the second system. The adjustment satisfies Conditions A(i), A(ii), and A(iii), and so if Condition B(i) holds the second system is at least as good as the first. Our definition of equivalence therefore implies that two equivalent systems of taxation will have the property that each is at least as good as the other.

It is useful to introduce algebraic notation for the portfolio adjustment described above, as well as to express the conditions in terms of formulas which can be used to test equivalence between systems of taxation. To this end, we write the investor portfolio adjustment as a matrix, $\mathbf{Q}$, with rows corresponding to investors, and we write the government portfolio adjustment as a matrix $\mathbf{Q}$. Both matrices have columns corresponding to assets.[15] The adjusted portfolios are thus $\mathbf{P} + \mathbf{Q}$ and $\mathbf{P}_G + \mathbf{Q}_G$. In the next few paragraphs we express the above conditions in terms of this new notation.

Condition A(i) requires that the elements of each row of $\mathbf{Q}$ must sum to zero, since this sum equals the change in initial wealth of the investors. Condition A(ii) requires that the elements of $\mathbf{Q}_G$ be equal to the sum of the columns of $\mathbf{Q}$. As a result of these two conditions, we see that these adjustment matrices must be of the form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v} & -\mathbf{v} \end{bmatrix} \quad \text{and} \quad \mathbf{Q}_G = \begin{bmatrix} -\mathbf{1}^t\mathbf{v} & \mathbf{1}^t\mathbf{v} \end{bmatrix}, \tag{4}$$

where $\mathbf{v}$ is a vector representing the change in asset holdings for each individual investor, and where $\mathbf{1}^t\mathbf{v}$ denotes the sum of all these changes for individual investors.[16] Equation (4) shows that the $\mathbf{v}$ adjustment of individuals in risky asset holdings must be offset by the $-\mathbf{v}$ adjustment in risk-free asset holdings, so that increased stock holdings, for example, are

[13] This follows because the government takes offsetting adjustment positions relative to the investors.

[14] This is true for investors because of Condition A(iii). It is true for the government because the change for the government is equal to the aggregate of the changes for each individual investor, since the government has taken offsetting adjustment positions relative to the investors and since the government naturally has an offsetting position relative to the tax paid by the individual investors.

[15] As with the $\mathbf{P}$ portfolios, the first column corresponds to the risky asset, and the second column corresponds to the risk-free asset.

[16] The superscript t denotes transposition of the vector of ones, $\mathbf{1}$, and this matrix product $\mathbf{1}^t\mathbf{v}$ is simply the sum of the elements of $\mathbf{v}$.

financed by debt.[17] The equation also shows that the government takes the other side of the changes investors require so that aggregate investment in each asset does not change.

Condition A(iii) can be written as

$$\mathbf{W}(\mathbf{P};\mathrm{T}) = \mathbf{W}(\mathbf{P}+\mathbf{Q};\mathrm{T}').$$

Using the definition of $\mathbf{W}$ in Equation (2) and simplifying, we find that this condition is equivalent to

$$\overbrace{\mathbf{P}(\mathbf{1}+\mathbf{r}) - (\mathbf{P}+\mathbf{Q})(\mathbf{1}+\mathbf{r})}^{\text{Pre-Tax Change in Return}} = \overbrace{\mathrm{T}(\mathbf{P},\mathbf{r}) - \mathrm{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})}^{\text{Change in Tax}}.$$

This can be simplified somewhat to the form From the discussion above, we know that the sum of the elements of each row of $\mathbf{Q}$ must be zero, and this means that $\mathbf{Q1}$, which is the vector of such sums, must be equal to a vector of zeros. As a result, Condition A(iii) states simply that

$$\mathbf{Qr} = \mathrm{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}) - \mathrm{T}(\mathbf{P},\mathbf{r}), \tag{5}$$

which means that the return on the adjustment portfolio is equal to the difference in tax payable under the two regimes for every investor.

We can now summarize the foregoing discussion and restate the above four conditions for equivalence as two more compact conditions. To wit, T is equivalent to T′ if and only if the following two conditions hold:

C(i) for each portfolio allocation $\mathbf{P}$, there is a portfolio adjustment $\mathbf{Q}$ of the form specified in Equation (4) that satisfies Equation (5); and

C(ii) for each portfolio allocation $\mathbf{P}'$, there is a portfolio allocation $\mathbf{P}$ such that $\mathbf{P}' = \mathbf{P}+\mathbf{Q}$, where $\mathbf{Q}$ is the adjustment from Condition C(i).

If Condition C(i) is satisfied, then the government portfolio adjustment $\mathbf{Q}_G = -\mathbf{1}^t\mathbf{Q}$ can be made, and conditions A(i), A(ii), and A(iii) are all satisfied. Also, if condition C(ii) is satisfied, then Condition B(i) is satisfied. The conditions C(i) and C(ii) therefore imply the prior four conditions, and conversely the four earlier conditions imply C(i) and C(ii). These two conditions are therefore equivalent to the prior four.

2.3 Proportionate Taxation

We now derive the well-known result about the nature of proportionate taxes stated in the following theorem, due originally to Domar and Musgrave (1944, 1945) and, in the context of general equilibrium, to Kaplow (1991, 1994).

Theorem 1 (Domar-Musgrave-Kaplow) *A proportionate tax on investment income is equivalent, in the sense of Section 2.2, to a tax at the same rate levied on the risk-free return to capital.*

[17] A short position in the risk-free asset is simply debt.

Let $\mathbf{T}$ correspond to a proportionate tax on investment income, so that

$$\mathbf{T}(\mathbf{P}, \mathbf{r}) = t\mathbf{P}\mathbf{r},$$

where t is a constant tax rate. Let $\mathbf{T}'$ correspond to a tax at the same rate on only the risk-free rate of return to capital, so that

$$\mathbf{T}'(\mathbf{P}, \mathbf{r}) = tr_f\mathbf{P}\mathbf{1}.$$

In the case of this choice of $\mathbf{T}$ and $\mathbf{T}'$, Condition C(i) is satisfied if for any portfolio allocation choice $\mathbf{P}$ there is a portfolio adjustment $\mathbf{Q}$ of the form specified in Equation (4) and satisfying

$$\mathbf{Q}\mathbf{r} = tr_f\,(\mathbf{P} + \mathbf{Q})\,\mathbf{1} - t\mathbf{P}\mathbf{r} \tag{6}$$

With $\mathbf{a}$ and $\mathbf{v}$ as in the definitions of $\mathbf{P}$ and $\mathbf{Q}$ in Equations (1) and (4), we can simplify Equation (6) to

$$\mathbf{v}(x - r_f) = t\mathbf{a}(r_f - x), \tag{7}$$

where we have used the fact that $\mathbf{r}$ has the value given in Equation (3). The matrix

$$\mathbf{Q} = t\begin{bmatrix} -\mathbf{a} & \mathbf{a} \end{bmatrix} \tag{8}$$

therefore satisfies Equation (6), and it is also of the form required by Equation (4). Because such a $\mathbf{Q}$ can be found for any portfolio allocation choice $\mathbf{P}$, we see that Condition C(i) is satisfied.

We next determine whether any portfolio $\mathbf{P}'$ can be written as an adjusted portfolio $\mathbf{P} + \mathbf{Q}$ as required by Condition C(ii). For each portfolio $\mathbf{P}' = \begin{bmatrix} \mathbf{a}' & \mathbf{b}' \end{bmatrix}$, the adjusted portfolio corresponding to the portfolio

$$\mathbf{P} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a}' & \mathbf{b}' \end{bmatrix} + \frac{t}{1-t}\begin{bmatrix} \mathbf{a}' & -\mathbf{a}' \end{bmatrix} \tag{9}$$

is equal to $\mathbf{P}'$. That is, for this choice of $\mathbf{P}$, if $\mathbf{Q}$ is as in Equation (8), then $\mathbf{P} + \mathbf{Q} = \mathbf{P}'$, and so every $\mathbf{P}'$ can be written as $\mathbf{P} + \mathbf{Q}$ for some $\mathbf{P}$. Condition C(ii) is therefore satisfied.

Both conditions C(i) and C(ii) are satisfied, and so we conclude that a proportionate income tax is equivalent to a tax at the same rate on the risk-free rate of return to wealth.

2.4 A Numerical Example

In this Section, we illustrate the results of Section 2.3 with a numerical example.

We assume that the tax rate t of Section 2.3 is 20%, so that

$$\mathbf{T} = 20\% \times \mathbf{P}\mathbf{r} \quad \text{and} \quad \mathbf{T}' = 20\% \times r_f\mathbf{P}\mathbf{1}$$

are the proportionate tax and the tax on the risk-free return to capital, respectively. We assume that there is only one individual investor and that his portfolio under the proportionate tax consists of a \$100 investment in the risky asset and a \$0 investment in the risk-free asset. Thus,

$$\mathbf{P} = \begin{bmatrix} \$100 & \$0 \end{bmatrix}.$$

The final after-tax wealth of the investor with this portfolio under the proportionate tax system is

$$\begin{aligned}\mathbf{W}(\mathbf{P};\mathbf{T}) &= \$100 + \$100x \times (1 - 20\%) \\ &= \$100 + \$80x,\end{aligned}$$

where x is the stochastic return on the risky asset.

Since $\mathbf{a} = \$100$ and $t = 20\%$, we see that in this case the adjustment portfolio is $\mathbf{Q} = \begin{bmatrix}-\$20 & \$20\end{bmatrix}$, and the adjusted portfolio is

$$\mathbf{P}' = \mathbf{P} + \mathbf{Q} = \begin{bmatrix}\$80 & \$20\end{bmatrix},$$

and the final after-tax wealth of the investor with this portfolio under the tax on the risk-free return to wealth is

$$\begin{aligned}\mathbf{W}(\mathbf{P}';\mathbf{T}') &= \$80 \times (1 + x) + \$20 \times (1 + r_f) - \$100 \times r_f \times 20\% \\ &= \$100 + \$80x.\end{aligned}$$

As an alternative example, suppose that we start with a portfolio $\mathbf{P}' = \begin{bmatrix}\$100 & \$0\end{bmatrix}$. Then the equivalent $\mathbf{P}$ portfolio specified in Equation (9) is

$$\mathbf{P} = \begin{bmatrix}\$100 & \$0\end{bmatrix} + \frac{20\%}{1 - 20\%}\begin{bmatrix}\$100 & -\$100\end{bmatrix} = \begin{bmatrix}\$125 & -\$25\end{bmatrix}.$$

The after-tax wealth of $\mathbf{P}$ under the $\mathbf{T}$ tax is

$$\begin{aligned}\mathbf{W}(\mathbf{P};\mathbf{T}) &= \$100 + (\$125x - \$25r_f)(1 - 20\%) \\ &= \$100 + \$100x - \$20r_f,\end{aligned}$$

and this is the same as the after-tax wealth of $\mathbf{P}'$ under the $\mathbf{T}'$, namely

$$\begin{aligned}\mathbf{W}(\mathbf{P}';\mathbf{T}') &= \$100 + \$100x - 20\% \times \$100r_f \\ &= \$100 + \$100x - \$20r_f.\end{aligned}$$

3 General Systems of Taxation

In this Section we study a variety of systems of taxation using the model developed in Section 2. In Section 3.1, we consider a tax with no loss offset as an example of a non-proportionate tax, and we carry out the analysis under the simplifyiug assumption that the risky asset has only two possible levels of return. In Section 3.3, we extend our analysis to more general examples of non-proportionate taxes, keeping the simplifying assumption that the risky asset has binary return possibilities. In Section 3.4, we discuss generalizations of the analysis to situations in which the risky asset may have more than two possible levels of return.

3.1 Loss Disallowance with Binary Returns

We now turn to studying taxation which is neither proportionate nor equivalent to a proportionate system. For now, we deal just with the case in which only two levels of return are possible for the risky asset. This simplifying assumption makes the analysis more tractable while still allowing the fundamental ideas to be illustrated. We also make the analysis more manageable by limiting ourselves, for the time being, to the case of a tax that is proportionate for gains but has no allowance for loss deductions.

We denote the two values that the risky asset return can assume as x_u and x_d, and we assume that $x_u > r_f > x_d$, so that it is possible for the risky asset to do either better or worse than the risk-free asset. Figure 1 illustrates the possible values of x and the corresponding values of the return vector $\mathbf{r}$.

Figure 1: Binary possibilities for x and $\mathbf{r}$.

The tax system that taxes gains at a constant rate but does not allow any deduction for losses can be represented by the function

$$\mathbf{T}(\mathbf{P}, \mathbf{r}) = t \times \max(0, \mathbf{Pr}), \tag{10}$$

where t is the constant rate at which gains are taxed, and $\max(0, c)$ is equal to c when c is positive and is equal to zero otherwise.

To understand $\mathbf{T}$ better, we look for a different system of taxation to which it is equivalent. Ideally, we would like the equivalent tax to be easier to understand and compare to other taxes than the system $\mathbf{T}$. Obvious candidates for satisfying this criterion would be a proportionate tax or a tax on wealth, but we can see from the results and techniques of Section 2.3 that $\mathbf{T}$ is not equivalent to such taxes. There is another basic class of systems of taxation to which we can look, however: those systems that do not depend upon what level of return is eventually achieved by the risky asset. Remarkably, there is a tax in this class, $\mathbf{T}'$, that is equivalent to $\mathbf{T}$, and we spend the remainder of this section describing $\mathbf{T}'$ and demonstrating the equivalence. In Sections 3.3 and 3.4, we study further the extent to which general tax systems can be expressed in terms of systems of this basic class.

We need to define specifically what the system of tax, $\mathbf{T}'$, described in the preceding paragraph is, and we also need to prove that it is equivalent to $\mathbf{T}$. It is easiest to carry out these tasks in concert, using the property of $\mathbf{T}'$ that we require, namely that it does not depend upon the return outcome for the risky asset, and using the fact that Conditions C(i) and C(ii) must be met if $\mathbf{T}$ and $\mathbf{T}'$ are equivalent. Figure 2 illustrates the requirement of Condition C(i) in both possibilities for the risky asset return. Because we require that $\mathbf{T}'$ not depend upon which level of return is actually achieved by the risky asset, the two possibilities for $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$ must give the same answer. Equating these quantities and solving, we find that

$$\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}) = \frac{x_u\mathbf{T}(\mathbf{P},\mathbf{r}_d) - x_d\mathbf{T}(\mathbf{P},\mathbf{r}_u)}{x_u - x_d} + \frac{\mathbf{T}(\mathbf{P},\mathbf{r}_u) - \mathbf{T}(\mathbf{P},\mathbf{r}_d)}{x_u - x_d} r_f, \tag{11}$$

where

$$\mathbf{Q} = \begin{bmatrix}\mathbf{v} & -\mathbf{v}\end{bmatrix} \quad \text{and} \quad \mathbf{v} = \frac{\mathbf{T}(\mathbf{P},\mathbf{r}_d) - \mathbf{T}(\mathbf{P},\mathbf{r}_u)}{x_u - x_d}. \tag{12}$$

Equation (11) serves to define $\mathbf{T}'(\mathbf{P}',\mathbf{r})$ for $\mathbf{P}' = \mathbf{P}+\mathbf{Q}$, and it, combined with Equation (12), shows that Condition C(i) is satisfied.

$$\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}) \begin{cases} \mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}_u) = \mathbf{Q}\mathbf{r}_u + \mathbf{T}(\mathbf{P},\mathbf{r}_u) \\ \mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}_d) = \mathbf{Q}\mathbf{r}_d + \mathbf{T}(\mathbf{P},\mathbf{r}_d) \end{cases}$$

Figure 2: Binary possibilities for $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$, assuming Condition C(i) is satisfied.

It remains to prove that Condition C(ii) is satisfied. This condition holds if for every portfolio $\mathbf{P}'$ there is some portfolio $\mathbf{P}$ and a corresponding $\mathbf{Q}$, as defined in Equation (12), such that $\mathbf{P}' = \mathbf{P}+\mathbf{Q}$. It is possible to demonstrate that this condition is indeed satisfied, and the details of the necessary calculations, as well as formulas for $\mathbf{P}'$ in terms of $\mathbf{P}$ are provided in the Appendix.

We have now demonstrated that the tax which is proportionate on gains but disallows losses is equivalent to a tax which does not depend upon which return is realized by the risky asset, at least in the case of only a binary possibility for returns. We still have not, however, expressed this equivalent tax, $\mathbf{T}'$, in any manner other than by reference to the original tax, $\mathbf{T}$, as in Equation (11), for example. Perhaps the most interesting feature of $\mathbf{T}'$ is the form that it takes when expressed directly and without reference to $\mathbf{T}$, and this is the content of the following theorem.

Theorem 2 *Let $\mathbf{T}$ be the tax system that imposes a proportionate tax on gains at the rate t but does not allow any offsetting deduction for losses, and let x_u and x_d be as above. There is a system of taxation, $\mathbf{T}'$, that is equivalent to $\mathbf{T}$ but does not depend upon the return actually achieved by the risky asset. It is defined as follows. Let $\mathbf{P}' = \begin{bmatrix}\mathbf{a}' & \mathbf{b}'\end{bmatrix}$ be the portfolio choice matrix, so that $\begin{bmatrix}a_i' & b_i'\end{bmatrix}$ is the portfolio of the i-th investor, and let $w_i = a_i' + b_i'$ denote the initial wealth of the i-th investor. The tax owed by the i-th investor under $\mathbf{T}'$ is as follows.*

If the investor has a significant portion of his wealth in a long position in the asset, such that $a_i' > \frac{(1-t)w_i r_f}{r_f - x_d}$, then the tax due is

$$\mathbf{T}_i'(\mathbf{P}', \mathbf{r}) = \frac{\overbrace{ta_i'(x_u - r_f) +}^{\text{Tax on Risk}} \overbrace{tw_i r_f}^{\text{Tax on Wealth}}}{\underbrace{1 + (1-t)/D}_{\text{Adjustment Factor}}}.$$

Here we have used the abbreviation $D = \frac{r_f - x_d}{x_u - r_f}$, which will generally be close to one, provided the upside and downside risks are roughly balanced relative to the risk-free rate. If the investor has a short position in the asset that is significant compared to his wealth, such that $a_i' < -\frac{(1-t)w_i r_f}{x_u - r_f}$, then the tax due is

$$\mathbf{T}_i'(\mathbf{P}', \mathbf{r}) = \frac{\overbrace{-ta_i'(r_f - x_d) +}^{\text{Tax on Risk}} \overbrace{tw_i r_f}^{\text{Tax on Wealth}}}{\underbrace{1 + (1-t)D}_{\text{Adjustment Factor}}},$$

where D has the same definition as before. Finally, if the investor has a long or short position in the asset which is not very significant compared to his overall wealth, such that such that $-\frac{(1-t)w_i r_f}{x_u - r_f} < a_i' < \frac{(1-t)w_i r_f}{x_d - r_f}$, then the tax due is

$$\mathbf{T}_i'(\mathbf{P}', \mathbf{r}) = \overbrace{tw_i r_f}^{\text{Tax on Wealth}}.$$

The most striking aspect of the theorem is that although the tax does not depend upon the level of risky return actually achieved, it does depend on the range of risky returns possible. In particular, if an investor holds a long position in the risky asset, a component of his tax is proportional to the excess of the high potential risky return over the risk-free rate. In a complementary way, if an investor holds a short position in the risky asset, a component of his tax is proportional to the excess of the risk-free rate over the low potential risky return over the risk-free rate.[18] The remaining component, unrelated to risk, is equal to a tax on the risk-free return to initial wealth, w_i.

The theorem stands in sharp contrast to the conventional understanding that an income tax does not burden risky returns to assets. While this understanding may be true, under suitable assumptions, for proportionate taxes, it fails dramatically for non-proportionate taxes. Even one of the simplest non-proportionate taxes, a proportionate tax on gains with no offset for losses, directly burdens the risk, as expressed by the possible deviation from the risk-free rate of return.

3.2 A Numerical Example

To illustrate the mechanics of the results of the preceding section, we consider a particular numerical example.

[18] In this case, the tax owed on this component is still negative, since both a_i' and $(x_d - r_f)$ are negative, and their product is therefore positive.

Suppose that an investor's portfolio consists of \$100 entirely invested in the risky asset. Suppose further that $x_u = 15\%$, $x_d = -10\%$, $r_f = 5\%$, and the tax rate $t = 20\%$.

Under the **T** system, with proportionate taxation of gains and no loss offsets, the investor has final after-tax wealth of either \$112 or \$90, as illustrated in Figure 3.

$$\$100 \nearrow \quad \$100 + \overbrace{15\% \times \$100}^{\text{Gain}} - \overbrace{15\% \times \$100 \times 20\%}^{\text{Tax}} = \$112$$

$$\$100 \searrow \quad \$100 - \underbrace{10\% \times \$100}_{\text{Loss}} = \$90$$

Figure 3: Outcomes for the example portfolio **P** under the **T** system.

To determine the corresponding portfolio $\mathbf{P}'$ which produces the same results relative to the $\mathbf{T}'$ tax, we refer to the formula for a' in terms of a from Section A.1 of the Appendix, and we find that $a' = \$88$. Therefore we must have $b' = \$12$, since total wealth is unchanged. The amount of tax owed on this portfolio can be determined by reference to Theorem 2. Inserting the value of $a' = \$88$, we see that the tax owed is $\mathbf{T}'(\mathbf{P}', \mathbf{r}) = \$1.8$, and this tax is the same no matter what outcome obtains for the risky asset. Figure 4 illustrates the computation of tax and returns for the portfolio $\mathbf{P}'$ under the $\mathbf{T}'$ system of taxation, and the results of course match those in Figure 3.

$$\mathbf{P}' \nearrow \quad \$100 + \overbrace{15\% \times \$88 + 5\% \times \$12}^{\text{Gain}} - \overbrace{\$1.8}^{\text{Tax}} = \$112$$

$$\mathbf{P}' \searrow \quad \$100 - \underbrace{10\% \times \$88}_{\text{Loss}} + \underbrace{5\% \times \$12}_{\text{Gain}} - \underbrace{\$1.8}_{\text{Tax}} = \$90$$

Figure 4: Outcomes for the example portfolio $\mathbf{P}'$ under the $\mathbf{T}'$ system.

With the numbers in this example, we can also calculate the cut-off levels for the three different possible treatments described in Theorem 2. The quantity $\frac{(1-t)w_i r_f}{r_f - x_d}$ is equal to approximately \$26.67, and so when amounts in excess of this are held, the first treatment of the theorem is appropriate. In the example, the amount held is $a' = \$88$, and so the first treatment applies. The details of the calculation are illustrated in the equation:

$$\mathbf{T}'_i(\mathbf{P}', \mathbf{r}) = \frac{ta'_i(x_u - r_f) + tw_i r_f}{1 + (1-t)/D} = \frac{\overbrace{\$1.76}^{\text{Tax on Risk}} + \overbrace{\$1}^{\text{Tax on Wealth}}}{\underbrace{1.533}_{\text{Adjustment Factor}}} = \$1.80.$$

Notice that the risk tax component is more than 75% larger than the wealth tax component. Also, note that the adjustment factor in the denominator applies equally to both the tax on risk and the tax on wealth.

3.3 General Tax Systems with Binary Returns

In this section, we continue to assume that the return to the risky asset can take on only two possible values, but we now move to considering systems of taxation more general than simply the loss disallowance system of Section 3.1. Much of the work we have already done carries over to more general settings. We outline the general ideas here.

Let $\mathbf{T}$ be an arbitrary tax system. As in Section 3.1, we seek an equivalent system that is easier to understand and compare to other taxes. Following the tack taken in that section, we look for an equivalent tax $\mathbf{T}'$ of the class of tax systems that assign an amount of tax independent of the return value actually achieved by the risky asset. If such a $\mathbf{T}'$ equivalent to $\mathbf{T}$ exists, then Condition C(i) is satisfied and the two possible values for $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$ are as illustrated in Figure 2. Also, since $\mathbf{T}'$ does not depend upon the which return value is achieved by the risky asset, we can equate the two possible values for $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$ and solve to see that Equation (11) describes $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$ and Equation (12) describes $\mathbf{Q}$. For convenience of reference, we restate the results here:

$$\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r}) = \frac{x_u\mathbf{T}(\mathbf{P},\mathbf{r}_d) - x_d\mathbf{T}(\mathbf{P},\mathbf{r}_u)}{x_u - x_d} + \frac{\mathbf{T}(\mathbf{P},\mathbf{r}_u) - \mathbf{T}(\mathbf{P},\mathbf{r}_d)}{x_u - x_d} r_f, \tag{13}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathrm{v} & -\mathrm{v} \end{bmatrix} \quad \text{and} \quad \mathrm{v} = \frac{\mathbf{T}(\mathbf{P},\mathbf{r}_d) - \mathbf{T}(\mathbf{P},\mathbf{r}_u)}{x_u - x_d}. \tag{14}$$

Thus far, we have been able to use the work of Section 3.1 in order to determine the value of $\mathbf{T}'(\mathbf{P}+\mathbf{Q},\mathbf{r})$, for $\mathbf{Q}$ as in Equation (14), and to demonstrate that Condition C(i) is satisfied. It remains, however, to demonstrate that each portfolio $\mathbf{P}'$ can be written in the form $\mathbf{P}+\mathbf{Q}$ for some $\mathbf{P}$. Proving this will show that Condition C(ii) is satisfied and will also define the tax $\mathbf{T}'(\mathbf{P}',\mathbf{r})$ for every portfolio $\mathbf{P}'$. In Sections 3.1 and A.1, we were able to use the explicit form of the tax $\mathbf{T}$ in order to express each $\mathbf{P}'$ in terms of $\mathbf{P}+\mathbf{Q}$, but that is not possible here since the particular form of $\mathbf{T}$ is not specified. For a completely general form of the tax $\mathbf{T}$ it may in fact not be possible to obtain the desired result, but with some mild restrictions on the nature of $\mathbf{T}$, it is possible. The following theorem states the result.

Theorem 3 *Suppose that the tax system $\mathbf{T}$ is such that if the i-th taxpayer holds a_i units of the risky asset the difference between the tax he owes under the two possible scenarios for risky asset returns is bounded by a constant, $0 < c < 1$, times the absolute value of his position, multiplied by the difference in the possible risky returns. We can write this condition as a formula as well,*

$$|\mathbf{T}_i(\mathbf{P},\mathbf{r}_d) - \mathbf{T}_i(\mathbf{P},\mathbf{r}_u)| \leq c|a_i|(x_u - x_d). \tag{15}$$

If $\mathbf{T}$ satisfies this condition and is also a continuous function of its inputs, then it is possible for every $\mathbf{P}'$ to be written in the form $\mathbf{P}+\mathbf{Q}$ and so Equations (13) and (14) define the tax $\mathbf{T}'(\mathbf{P}',\mathbf{r})$ for all possible portfolios $\mathbf{P}'$ and $\mathbf{T}'$ is a tax which is equivalent to the $\mathbf{T}$ but does not depend on the return actually achieved by the risky asset.

Details of the proof of the theorem are given in Section A.3. It is useful to note that the conditions on $\mathbf{T}$ required by the theorem are not very limiting from a practical point of view. Equation 15 simply requires that tax burdens not change too drastically relative to the size of investment in an asset and the overall range of return levels. It will generally be satisfied if c is taken to be the highest marginal tax rate in the system. Also, the continuity of $\mathbf{T}$ is only required to ensure that erratic jumps in taxes payable don't occur in the system.

The theorem is an important theoretical result because it shows that the work of Section 3.1 extends to any reasonable system of taxation and is not special to a proportionate tax with a disallowance for losses. The only drawback is that no explicit form, similar to that provided in Theorem 2, is available for the tax $\mathbf{T}'$. This is an unavoidable result of the fact that Theorem 3 applies with generality to a wide range of tax systems $\mathbf{T}$ and no general explicit formula is possible. In the case of any particular tax system under consideration, however, an explicit form for the equivalent tax $\mathbf{T}'$ can be derived.

3.4 Options and Continuous Trading

Sections 3.1 and 3.3 provided powerful results showing that it is in general possible to transform a system of tax into an equivalent system which does not vary with the realized risky asset returns. Equivalences of this type represent a way to place various tax systems on a common footing so that their relative effects can be compared. An important limitation of the preceding sections, however, is of course the fact that the results are limited to the situation in which there is only a binary choice for the return to the risky asset. In this section, we discuss approaches for generalizing beyond this limitation.

The idea of the binary model in the preceding sections was that it was possible in such a simplified world to form perfect hedges of the option-like non-proportionate tax obligation in the general equilibrium framework we had developed. The cost of these hedges corresponds to the burden of the tax on risky asset returns.

In a more complex world, with more than two possible return levels for the risky asset, an investor could again imagine forming a hedge for his tax liability, and the cost of such a hedge would again represent the burden of the tax on risky asset returns. In a single time period model, however, if the investor is only able to take positions in the underlying risky asset, and if that asset has more than two possible return levels, it is not generally possible to hedge the effect of a non-proportionate tax. It would be possible, however, if instead of a single period in which to act the investor could trade continuously throughout the time horizon of the model. In this case, the investor could form a dynamic portfolio that evolved to hedge the tax obligation on an ongoing basis throughout the entire period.

There are difficulties with introducing dynamic trading. If the investor is allowed to trade dynamically, he will likely choose not only to update his hedge but also to update his investment positions as time evolves. To model such behavior, a multi-period general equilibrium model would be necessary. Such a model would be substantially more realistic and practical than a single period model and would capture well the possibility of dynamically hedging one's tax obligations. However, it would also be substantially more difficult to specify such a model and derive informative theoretical information from it. Nonetheless, the approach of working in a multi-period framework offers many benefits, and pursuing that avenue may be the best way to move forward and extend the results presented here.

In lieu of a general multi-period model, there is an alternative short-cut approach for dealing with more than two risky asset values. We could posit again a single-period model in which investors are only able to make investment decisions at the beginning of a period and must remain invested through the end of the period. However, we could add a class of special actors, called intermediaries, who unlike investors would trade continuously throughout the period and could create the dynamic hedges for tax obligations. The price the intermediaries would charge would be idealized to be equal to the present value of the cost of hedging, the same cost that the investors would incur themselves if they were able to trade continuously.

This short-cut is attractive because it does not require extension to a multi-period general equilibrium model. It is also realistic in some sense, since it is a common function of financial intermediaries to provide structured option-like products to clients and eliminate risk in the position through dynamic hedging. It is limited, however, to the extent that it may be unrealistic to treat investors as non-trading actors throughout the entire period while there are intermediaries who trade continuously. Nevertheless, the approach of trading intermediaries provides a straightforward mechanism for extending the model developed in the preceding sections from binary return possibilities to continuous ones. If this approach is taken, the results all carry over in general, and non-proportionate taxes are seen to burden risky asset returns just as option prices incorporate the risk of underlying assets.

4 Conclusion

Our results show that, unlike proportionate taxes, non-proportionate tax systems generally burden risky asset returns. We have seen this in detail in the case of a proportionate tax with no loss offset when the risky asset has only a binary choice of return levels. Our work shows that this type of result extends to much more general systems of taxation, and a broadening of the model to include continuously-trading intermediaries would operate to remove the restriction of only a binary choice for risky asset return levels. We thus conclude that, in contrast to the well-established thinking about proportionate taxes, non-proportionate taxes can and do burden risky returns.

The point that non-proportionate taxes behave differently from proportionate taxes is not a new one, nor is the point that such taxes are not covered by the arguments usually made about proportionate taxes. What is new, however, is the brining to bear on the question of non-proportionate taxes the sophisticated machinery typically used to study proportionate taxes. By analyzing non-proportionate taxes in a general equilibrium framework, we have demonstrated that the burden they impose on risk is specific, substantial and quantifiable.

These results are important, as is the new methodology used to obtain them. In the ongoing debate about what system of taxation should best be employed, it is critical to understand what systems of taxation are able to accomplish, both in theory and in practice. For example, the results of this paper show that an income tax with periodic mark-to-market requirements and loss disallowance rules is capable of burdening risky returns to assets and in fact does so in a concrete and calculable way.[19] Previously, this result may not have been obvious, or the burden might have been thought not to be so specific and quantifiable.

Some will view the information that a non-proportionate income tax can burden risky returns in a precise way as a clear indication that we should not have a non-proportionate income tax, or perhaps as further evidence that there should be no income tax at all. Others may view the ability of a non-proportionate income tax to burden risky returns as a strong reason for supporting some form of such a tax. The goal of the current paper is not to engage in this debate but rather to provide the tools, results, and ideas necessary for the issues to be well-defined and for the debate to be well-informed.

[19]We add the condition of period mark-to-market in order to avoid the issue of potentially unlimited deferral that can occur in a realization-based system. Such an issue is of course of great importance, but it is analytically separate from the question of whether an income tax can burden risky returns, over a specified time horizon during which all gains and losses will be realized or otherwise marked to market.

A Appendix

A.1 Demonstration that Condition C(ii) Holds in Section 3.1

Let $\mathbf{T}$ be as defined in Equation (10), and let $\mathbf{Q}$ and $\mathbf{v}$ be as defined in Equation (12). In this section of the appendix, we demonstrate the result relied upon in Section 3.1 that each portfolio $\mathbf{P}' = \begin{bmatrix}\mathbf{a}' & \mathbf{b}'\end{bmatrix}$, can be written as $\mathbf{P}+\mathbf{Q}$, where $\mathbf{P} = \begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix}\mathbf{v} & -\mathbf{v}\end{bmatrix}$.

To start, notice that, since $\mathbf{a}'+\mathbf{b}' = \mathbf{a}+\mathbf{b}$, we must have

$$\mathbf{b} = \mathbf{a}'+\mathbf{b}'-\mathbf{a},$$

and so it is necessary only to find a value of $\mathbf{a}$ for which $\mathbf{P}' = \mathbf{P}+\mathbf{Q}$.

For each investor i, we consider separately the i-th element of $\mathbf{a}$, denoted a_i, and similarly for $\mathbf{b}$, $\mathbf{a}'$ and $\mathbf{v}$. Each a_i must satisfy

$$a_i' = a_i + v_i = a_i + t\frac{\max(0, a_i x_d + b_i r_f) - \max(0, a_i x_u + b_i r_f)}{x_u - x_d}.$$

Since $b_i = a_i' + b_i' - a_i$, this is the same as

$$a_i' = a_i + t\frac{\max(0, a_i(x_d - r_f) + w_i r_f) - \max(0, a_i(x_u - r_f) + w_i r_f)}{x_u - x_d},$$

where $w_i = a_i' + b_i'$ is the initial wealth of the i-th investor. The first max function in the numerator is positive whenever $a_i < -\frac{w_i r_f}{x_d - r_f}$, and this upper bound is a positive number since $w_i, r_f > 0$ and $r_f > x_d$. The second max function in the numerator is positive whenever $a_i > -\frac{w_i r_f}{x_u - r_f}$, and this lower bound is a negative number, since $x_u > r_f$. We therefore see that

$$a_i' = \begin{cases} \frac{x_u-(1-t)x_d - tr_f}{x_u - x_d} a_i + t\frac{w_i}{x_u - x_d} r_f, & \text{if } a_i < -\frac{w_i r_f}{x_u - r_f}; \\ a_i(1-t), & \text{if } -\frac{w_i r_f}{x_u - r_f} \le a_i \le -\frac{w_i r_f}{x_d - r_f}; \text{ and} \\ \frac{tr_f + (1-t)x_u - x_d}{x_u - x_d} a_i - t\frac{w_i}{x_u - x_d} r_f, & \text{if } -\frac{w_i r_f}{x_d - r_f} < a_i. \end{cases}$$

This relationship can be inverted to find that

$$a_i = \begin{cases} \frac{x_u - x_d}{x_u - (1-t)x_d - tr_f} a_i' - \frac{tw_i}{x_u-(1-t)x_d - tr_f} r_f, & \text{if } a_i' < -\frac{(1-t)w_i r_f}{x_u - r_f}; \\ \frac{1}{1-t} a_i' & \text{if } -\frac{(1-t)w_i r_f}{x_u - r_f} < a_i' < -\frac{(1-t)w_i r_f}{x_d - r_f}; \text{ and} \\ \frac{x_u - x_d}{tr_f + (1-t)x_u - x_d} a_i' + \frac{tw_i}{tr_f + (1-t)x_u - x_d} r_f & \text{if } -\frac{(1-t)w_i r_f}{x_d - r_f} < a_i'. \end{cases} \tag{16}$$

This formula for the elements of $\mathbf{a}$ in terms of those of $\mathbf{a}'$, along with the relationship $\mathbf{b} = \mathbf{a}'+\mathbf{b}'-\mathbf{a}$ provides the mechanism through which $\mathbf{P}$ may be expressed in terms of $\mathbf{P}+\mathbf{Q}$, and so Condition C(ii) is seen to hold as required in Section 3.1.

A.2 Proof of Theorem 2

The equivalence between $\mathbf{T}$ and $\mathbf{T}'$ is established in Sections 3.1 and A.1. It remains only to demonstrate that the form of $\mathbf{T}'$ is as asserted in the theorem.

From Equation (11) we see that the i-th component of $\mathbf{T}'$ is

$$\mathbf{T}'_i(\mathbf{P}',\mathbf{r}) = t\frac{x_u \max(0, a_i(x_d - r_f) + w_i r_f) - x_d \max(0, a_i(x_u - r_f) + w_i r_f)}{x_u - x_d} - t\frac{\max(0, a_i(x_d - r_f) + w_i r_f) - \max(0, a_i(x_u - r_f) + w_i r_f)}{x_u - x_d} r_f,$$

where a_i is defined in terms of a'_i as in Equation (16), and where $w_i = a'_i + b'_i$ is the initial wealth of the investor. The three cases in Equation (16) exactly determine which of the max functions are positive and which are zero. In the first case, the first max function in each numerator is positive, in the second case, all four max functions are positive, and in the third case, only the last max function in each numerator is positive. Thus we find that

$$\mathbf{T}'_i(\mathbf{P}',\mathbf{r}) = \begin{cases} t(a_i(x_d - r_f) + w_i r_f)\left(\frac{x_u - r_f}{x_u - x_d}\right), & \text{if } a'_i < -\frac{(1-t)w_i r_f}{x_u - r_f}; \\ t w_i r_f, & \text{if } -\frac{(1-t)w_i r_f}{x_u - r_f} < a'_i < -\frac{(1-t)w_i r_f}{x_d - r_f}; \\ t(a_i(x_u - r_f) + w_i r_f)\left(\frac{r_f - x_d}{x_u - x_d}\right), & \text{if } -\frac{(1-t)w_i r_f}{x_d - r_f} < a'_i; \end{cases} \tag{17}$$

If we substitute the value of a_i in terms of a'_i from Equation (16) into this last equation, we find the values stated in the theorem.

A.3 Proof of Theorem 3

To prove Theorem 3, we need to show that when the requirements of the theorem are met, there is a $\mathbf{P}$ such that $\mathbf{P}' = \mathbf{P} + \mathbf{Q}$, where $\mathbf{Q}$ is as defined in Equation (14). We write

$$\mathbf{P}' = \begin{bmatrix}\mathbf{a}' & \mathbf{b}'\end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix}\mathbf{v} & -\mathbf{v}\end{bmatrix}.$$

Note that since $\mathbf{Q}$ does not change the wealth of any investor, the $\mathbf{b}$ is determined by $\mathbf{a}$. More specifically,

$$\mathbf{b} = \mathbf{b}' + \mathbf{a}' - \mathbf{a}.$$

Thus, we need only determine whether it is possible to solve for $\mathbf{a}$, and we approach this problem one element of $\mathbf{a}$ at a time.

For the i-th investor, we wish to find an a_i which satisfies

$$a'_i = a_i + v_i, \quad \text{where} \quad v_i = \frac{\mathbf{T}_i(\mathbf{P},\mathbf{r}_d) - \mathbf{T}_i(\mathbf{P},\mathbf{r}_u)}{x_u - x_d}.$$

From the requirement of the theorem, we see that $|v_i| \le c|a_i|$ for some fixed constant $0 < c < 1$. As a result the quantity $a_i + v_i$ takes on arbitrarily large positive values as a_i takes on arbitrarily large positive values, and similarly with arbitrarily large negative values. The continuity of $\mathbf{T}$ required by the theorem implies that $a_i + v_i$ varies continuously as well, and so we see that every real value must be assumed for some a_i by the function $a_i + v_i$. In particular, the value a'_i must be realized for some value of a_i. This means that there is a solution a_i for the i-th investor, and aggregating the result over all investors, we see that there is a solution $\mathbf{a}$ to the equation $\mathbf{a}' = \mathbf{a} + \mathbf{v}$, which proves the theorem.

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